prof H'm JIBRIL

18|1|2022

MATHS 401 (THEORY OF ORDINARY DIFFERENTIAL EQUATION)

COURSE OUTLINE

1. Gamma and Beta functions :- Definitions and relationship between Gamma and Beta functions, Evaluation of Integrals (using Gamma and Beta functions).
2. Sturm - Liouville problem :- orthogonal and orthonormal functions, solutions of the S-L Equation.
3. Ordinary Differential Equations: Series solutions of second order linear differential equations: ordinary and singular point(s), frobenius method etc.
4. solutions of Legendre and Bessels function: Legendre and Bessels differentials Equations, Bessels functions of the first and second Kind etc. Generating functions, Recurrence relations etc.
5. Theory of solutions of Initial value problem (I.V.P)
6. Integral Equation: Equations with separable Kernels, solutions of Integral Equations, Classification Volterra and Fredholm types, solutions using Laplace and fourier transform. Reduction of an O.D.E to an Integral Equation, stability, Lyapunov function. Symmetric Kernels etc.

REFERENCE

i DASS HK

ii - STROUD K.A 4th

iii - SPIEGEL M.R

iv - Schaum's outline series

HK 98 1174

GAMMA AND BETA FUNCTIONS:-

GAMMA FUNCTIONS:- The function denoted by

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\,dx \qquad n>0$$

$$\therefore \Gamma n = \lim_{m\to\infty} \int_0^m x^{n-1} e^{-x}\,dx.$$

IMPROPER INTEGRALS: An Integral of $\int_a^b f(x)\,dx$ is said to be Improper if

i. One of the limits of Integration is Infinite, Examples

$$\int_0^\infty \frac{1}{x^2+1}\,dx, \quad \int_{-\infty}^{2} (x+1)\,dx, \quad \int_{-\infty}^{\infty} \cos \pi x\,dx$$

ii. $f(x)$ is discontinuous at $x \in [a,b]$ Example

$$\int_1^2 \frac{1}{x-1}\,dx, \quad \int_0^{\pi/2} \cot x\,dx \text{ etc.}$$

The Gamma function (Γ of n) exist if the Integral above Converges. If however the Integral diverges then the Gamma functions does not exist.

Examples: $\Gamma 2 = \int_0^\infty x e^{-x}\,dx = \lim_{m\to\infty} \int_0^m x e^{-x}\,dx$

$$= \lim_{m\to\infty} \left[-x e^{-x}\Big|_0^m + \int_0^m e^{-x}\,dx\right]$$

$$= \lim_{m\to\infty} \left[-m e^{-m} - e^{-m} + 1\right]$$

$$= 1 \quad \therefore \Gamma 2 = 1! = 1$$

NOTE

for $n<0$: $\Gamma n = \frac{\Gamma(n+1)}{n}$

for $n>0$: $\Gamma(n+1) = n\Gamma n$

$\frac{1}{2}! = \Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2}$

$\frac{3}{2}! = \Gamma(\frac{3}{2}+1) = \Gamma\frac{5}{2} = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$\frac{5}{2}! = \Gamma(\frac{5}{2}+1) = \Gamma\frac{7}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$\int u\,dv = uv - \int v\,du$

$x = u \Rightarrow du = dx$

$dv = e^{-x}$

$v = -e^{-x}$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = \lim_{m\to\infty}\int_0^m x^n e^{-x}\,dx$$

$$= \lim_{m\to\infty}\left[-x^n e^{-x}\Big|_0^m + n\int_0^m x^{n-1}e^{-x}\,dx\right]$$

$$= n\int_0^\infty x^{n-1}e^{-x}\,dx = n\Gamma(n)$$

$$\therefore \Gamma(n+1) = n\Gamma(n)$$

$\Gamma(n+1) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) \ldots\ldots n(n-1)(n-2)\ldots\Gamma(1)$

or $\Gamma(n+1) = n(n-1)(n-2)\ldots 1 = n!$

$\therefore \Gamma(n+1) = n!$

Therefore, gamma function is called the factorial function.

Example $\Gamma(-2) = \int_0^\infty x^{-3}e^{-x}\,dx$ — diverges

Exercise Evaluate (i) $\Gamma(10)$ (ii) $\Gamma(5)$ (iii) $\Gamma(-2)$

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Examples 1) Evaluate $\int_0^\infty x^6 e^{-3x}\,dx$

Soln: let $u = 3x \Rightarrow du = 3dx \therefore dx = \frac{du}{3}$ & $x = u/3$

$$\therefore I = \int_0^\infty (u/3)^6 e^{-u}\cdot du/3 = \int_0^\infty \frac{1}{3^6}\cdot u^6 e^{-u}\cdot\frac{du}{3}$$

$$= \frac{1}{3^7}\int_0^\infty u^6 e^{-u}\,du = \frac{1}{3^7}\Gamma(7) = \frac{1}{3^7}\cdot 6!$$

$$= \frac{6!}{2187} = \frac{720}{2187} = \frac{80}{243}$$

2) Evaluate $\Gamma(1/2) = \int_0^\infty x^{-1/2}e^{-x}\,dx$

let $x = y^2 \Rightarrow dx = 2y\,dy$

$$\int_0^\infty (y^2)^{-1/2} e^{-y^2} \cdot 2y\,dy = 2\int_0^\infty e^{-y^2}\,dy$$

let if $x = z^2 \Rightarrow \Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-z^2}\,dz$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \left(2\int_0^\infty e^{-y^2}dy\right)\left(2\int_0^\infty e^{-z^2}dz\right) = 4\int_0^\infty\!\!\int_0^\infty e^{-(y^2+z^2)}\,dy\,dz$$

let $y = r\sin\theta$, $z = r\cos\theta$, $0 \le r \le \infty$, $0 \le \theta \le \pi/2$

Now $\left(\Gamma\tfrac{1}{2}\right)^2 = 4\int_{\theta=0}^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta = 4\int_0 r\,e^{-r^2}\int_{\theta=0}^{\pi/2} d\theta$

~~Now from~~ $\int r e^{-r^2}\,dr$

let $u = r^2$ $\quad du = 2r\,dr \quad dr = \frac{du}{2r}$

$$\int_0^\infty r e^{-r^2}dr = \int_0^\infty r\cdot e^{-u}\cdot\frac{du}{2r} = \frac{1}{2}\int_0^\infty e^{-u}\,du = \frac{1}{2}\left[-e^{-u}\right]_0^\infty = 1/2$$

now $4\int_0^\infty r e^{-r^2}dr\int_0^{\pi/2}d\theta$ but $\int r e^{-r^2}dr = 1/2$

$$\Rightarrow 4\int_0^\infty r e^{-r^2}dr\int_{\theta=0}^{\pi/2}d\theta = 4\cdot\tfrac{1}{2}\int_0^{\pi/2}d\theta = 2\left[\theta\right]_0^{\pi/2}$$

$$2\left[\pi/2 - 0\right] = 2(\pi/2) = \pi$$

$\therefore \left(\Gamma\tfrac{1}{2}\right)^2 = \sqrt{\pi}$

Example Evaluate $\int_0^\infty x^{3/2}e^{-x}\,dx$

$$= \Gamma\tfrac{5}{2} = \tfrac{3}{2}\Gamma\tfrac{3}{2} = \tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma\tfrac{1}{2} = 3\sqrt{\pi}/4$$

Example Evaluate $\int_0^\infty x^{-3/2}e^{-x}\,dx$

$$= \frac{\Gamma(-3/2+1)}{-3/2} = \frac{\Gamma(-1/2)}{-3/2} = \frac{\Gamma(-1/2+1)}{-3/2 - 1/2} = \frac{\Gamma\frac{1}{2}}{3/4} = \frac{\sqrt{\pi}}{3/4} = \frac{4\sqrt{\pi}}{3}$$

$$= \frac{\Gamma\frac{1}{2}}{(-3/2)(-1/2)} = \frac{4\sqrt{\pi}}{3}$$

BETA FUNCTION: The function of m, n is defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad, m,n>0 \quad \text{---} \quad (*)$$

let $x = \sin^2\theta$, $x=0 \Rightarrow \sin^2\theta = 0$ or $\sin\theta = 0$ or $\theta = 0$

When $dx = 2\sin\theta\cos\theta\,d\theta$, $x=1 \Rightarrow \sin^2\theta = 1$ or $\sin\theta = 1 \therefore \theta = \pi/2$

$$B(m,n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$\boxed{= 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta \quad, m,n>0 \quad \text{When } \cos^2\theta = 1-\sin^2\theta}$$

NB: $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$, $m+n \neq 0$.

Example Evaluate the integral

i) $\int_0^1 x^3(1-x)^2\,dx = B(4,3) = \dfrac{\Gamma 4\,\Gamma 3}{\Gamma 7}$

soln ~~let $x = 3u$~~ $\Rightarrow$ ~~$dx = 3du$~~

~~When $x=0$, $u=0$~~, ~~When~~ $= \dfrac{3!\,2!}{6!} = \dfrac{6\cdot 2}{6\times5\times4\times3\times2} = \dfrac{1}{60}$

ii) $\int_{\theta=0}^{\pi/2} \sin^5\theta\cos^3\theta\,d\theta = B(3,2)$

$= \dfrac{1}{2}\cdot\dfrac{\Gamma 3\cdot\Gamma 2}{\Gamma 5} = \dfrac{1}{2}\cdot\dfrac{\Gamma 3\,\Gamma 2}{\Gamma 5} = \dfrac{1}{2}\left(\dfrac{2!\,1!}{4!}\right) = \dfrac{1}{2}\left(\dfrac{2}{24}\right) = 1/24$

Exercise: Evaluate i) $\Gamma -5$ ii) $\Gamma 7/2$ (iii) $\Gamma -5/2$

NB: i $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$, $m+n \neq 0$.

ii - $B(m,n) = B(n,m)$:

proof: $\Gamma m = \int_0^\infty x^{m-1}e^{-x}\,dx$, let $x = u^2$

$\Rightarrow \Gamma m = \int_0^\infty u^{2m-2}\cdot 2u\,du \Rightarrow \Gamma m = 2\int_0^\infty u^{2m-1}e^{-u^2}\,du$

$\Gamma n = \int_0 x^{n-1} e^{-x} dx$, let $x = v^2$, $dx = 2v\,dv \Rightarrow$

$\Gamma n = \int_0^\infty v^{2n-2}\, 2v\,dv \Rightarrow \Gamma n = 2\int_0^\infty v^{2n-1} \cdot e^{-v^2} dv$

$\Rightarrow \Gamma m \Gamma n = 4\int_0^\infty \int_0^\infty u^{2m-1} \cdot v^{2n-1} e^{-(u^2+v^2)} du\,dv$

let $u = r\cos\theta$, $v = r\sin\theta$, $0 \le r < \infty$, $0 \le \theta \le \pi/2$.

$$\Rightarrow \Gamma m \Gamma n = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} (r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1} e^{-r^2} \cdot dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} r^{2m+2n-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, e^{-r^2} dr\,d\theta.$$

$$= \underbrace{2\int_{\theta=0}^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta}_{B(m,n)} \left(2\int_{r=0}^{\infty} r^{2m+2n-1} e^{-r^2} dr\right)$$

let $r^2 = y \Rightarrow r = y^{1/2}$

$$2\int_{r=0}^{\infty} r^{2m+2n-1} e^{-r^2} dr = \cancel{2}\int_{y=0}^{\infty} y^{1/2(2m+2n-1)} e^{-y} \cdot \frac{dy}{\cancel{2}y^{1/2}}$$

$$= \int_0^\infty y^{m+n-1} e^{-y} dy = \Gamma m+n.$$

$\therefore \Gamma m \Gamma n = B(m,n) \cdot \Gamma m+n$.

Hence $B(m,n) = \dfrac{\Gamma m \Gamma n}{\Gamma m+n}$ //

ii) $B(m,n) = B(n,m)$

proof: from $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

let $x = 1-u$

$\Rightarrow B(m,n) = \int_{u=1}^{0} (1-u)^{m-1} \cdot u^{n-1} \cdot (+du)$

$= \int_{u=0}^{1} u^{n-1}(1-u)^{m-1} du = B(n,m)$

$\Rightarrow B(m,n) = B(n,m)$ □

$\boxed{\text{Exam } \int_0^\infty\int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} \cdot y^{2n-1} dx\,dy}$

2 STURM - LOUVILLE PROBLEM 25/1/22

a) Initial value problem: An I.V.P is an ordinary differential equation together with the associated initial conditions. Examples

$$\frac{dy}{dx} = x+y, \quad y(0) = 1, \quad \frac{2d^2y}{dx^2} - x\frac{dy}{dx} + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

b) Boundary value problem: A B.V.P is a differential equation together with the associated boundary conditions. Examples

i. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(0) = 0, \quad y(\pi) = 0, \quad 0 \le x \le \pi$

ii. $\frac{\partial u}{\partial t} = \frac{2\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(2,t) = 0, \quad u(x,0) = \sin 3\pi x, \quad 0 \le x \le 2, \quad t > 0.$

c) Sturm Louville Equation:- The Sturm-Louville problem is a differential equation of the form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \left(q(x) + \lambda r(x)\right)y = 0$$

$$a_1 y(a) + a_2 y'(a) = 0, \quad b_1 y(b) + b_2 y'(b) = 0, \quad a \le x \le b, \quad p(x) \ne 0$$

$$p(x)\frac{d^2y}{dx^2} + p'(x)\frac{dy}{dx} + \left(q(x) + \lambda r(x)\right)y = 0$$

Where λ is a parameter $r(x)$ is called the density function. The above differential equation has solutions for some value of λ. These values ($\lambda = \lambda_1, \lambda_2 \ldots \lambda_n$) are called the eigen values of the problem.

for each eigenvalue there is a corresponding ~~eigen~~ ~~functions of the~~ solution for the differential equation those solutions are called the eigen functions of the problem.

The eigenfunctions of the Sturm-Liouville problem i.e $\{y_n = [y_1(x), y_2(x) --- y_n(x)\}$ form an orthogonal and orthonormal set.

NB: Two functions $y_n(x)$ and $y_m(x)$ are said to be orthogonal if $\int_a^b y_n(x) \cdot r(x) \cdot y_m(x)\, dx = 0$

and are orthonormal if $\int_a^b y_n^2(x)\, dx = 1$

$$\int_a^b y_n(x) \cdot r(x) \cdot y_m(x)\, dx = \delta = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

δ ← Dirac delta Kronecker

Example: Show that the equation:

$y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0,\ 0 \leq x \leq \pi$ is a Sturm Liouville ~~problem~~. Hence solve the equation to determine the eigen value and the corresponding eigenfunctions. Show that the set of eigenfunctions forms an orthogonal and orthonormal set.

Soln

$y'' + \lambda y = 0,\quad y(0) = 0,\quad y(\pi) = 0$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0, \quad \begin{aligned} a_1 y(a) + a_2 y'(a) &= 0 \\ b_1 y(b) + b_2 y'(b) &= 0 \end{aligned}$$

$a \leq x \leq b.\quad (y')' + (0 + \lambda \cdot 1)y = 0$

$p(x) = 1,\ q(x) = 0,\ r(x) = 1,\ a = 0,\ b = \pi,\ a_1 = 1,\ a_2 = 0,\ b_1 = 1,\ b_2 = 0$

$y'' + \lambda y = 0$ let $y = e^{mx}$ $\therefore m^2 + \lambda = 0$

$m = \pm\sqrt{\lambda}\, i$. The general solution to the equation is

$y(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$y(0) = 0 \Rightarrow A\cos\sqrt{\lambda}\cdot 0 + B\sin\sqrt{\lambda}\cdot 0$

$\Rightarrow 0 = A\cdot 1 \quad \therefore A = 0$

for $y(x) = B\sin\sqrt{\lambda}\,x$

again $y(\pi) = 0 \Rightarrow B\sin\sqrt{\lambda}\pi = 0 \therefore \sin\sqrt{\lambda}\pi = 0$

$\therefore \sqrt{\lambda}\pi = n\pi, \quad n = 0, 1, 2, 3 \ldots$

$\sqrt{\lambda} = n \therefore \lambda = n^2 \{n = 0, 1, 2, 3 \ldots$

Thus, $\lambda_n = n^2 = \{0, 1, 4, \cancel{9}, 16 \ldots$

is the set of the eigen values of the problem.

$y(x) = B\sin nx$ or $y_n(x) = B_n\sin nx, \; n = 0, 1, 2 \ldots$

$\therefore$ The set of eigenfunctions

$y_n(x) = \{0, B_1\sin x, B_2\sin 2x, \ldots$

To show that the eigen functions forms an orthogonal set

$$\int_{x=a}^{b} y_n(x)\, r(x)\, y_m(x)\, dx = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}, \quad r(x) = 1$$

$$\Rightarrow \int_0^\pi B_n\sin nx \cdot B_m\sin mx\, dx = 0$$

$$\Rightarrow B_n B_m \int_0^\pi \sin nx \sin mx\, dx = \frac{B_n B_m}{2}\int_0^\pi \cos(n-m)x - \cos(n+m)x\, dx$$

$$\Rightarrow B_n B_m \left[\frac{\sin(n-m)x}{n-m}\Big|_0^\pi - \frac{\sin(n+m)x}{n+m}\Big|_0^\pi \right]$$

$$\Rightarrow B_n B_m \left[\frac{\sin(n-m)\pi}{n-m} - \frac{\sin(n+m)\pi}{n+m} \right] = 0$$

$\therefore \int_0^\pi y_n \cdot y_m\, dx = 0$ - The set is orthogonal.

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

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$\int_a^b y_n(x)\cdot r(x)\cdot y_m(x)\, dx = 0$ — orthogonal

$= 1$ — orthonormal

$\delta_{m,n} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$

$\cos 2\theta = 1 - 2\sin^2\theta$
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

To show that the set of eigenfunctions forms an orthonormal

$\therefore \int_{x=a}^{b} y_n^2 \, dx = 1 \Rightarrow \int_0^{\pi} B_n^2 \sin^2 nx \, dx = 1$

But $\int_0^{\pi} B_n^2 \sin^2 nx \, dx = \frac{B_n^2}{2}\int_0^{\pi}(1-\cos 2nx)\, dx = 1$

$\therefore \frac{B_n^2}{2}\left(x - \frac{\sin 2nx}{2n}\right)\Big|_0^{\pi} = 1 \Rightarrow \frac{B_n^2}{2}\left(\pi - \underbrace{\sin 2n\pi}_{=0}\right) = 1$

Or $\frac{B_n^2}{2}\cdot \pi = 1 \therefore B_n^2 = \frac{2}{\pi}$ i.e

$B_n = \sqrt{\frac{2}{\pi}}$ Normalisation Constant.

Orthogonal: $y_n(x) = \sqrt{\frac{2}{\pi}} \sin nx \Rightarrow y_n(x) = \{0, \sqrt{\frac{2}{\pi}}\sin x, \sqrt{\frac{2}{\pi}}\sin 2x \ldots\ldots\}$

Example 2 Show that the Equation:

$(xy')' + \lambda/x \, y = 0, \quad y'(1) = 0, \; y(e) = 0, \quad 1 \le x \le e$

is a Sturm - Liouville problem.

Hence solve the Eqn + determine the eigenvalues and the corresponding eigenfunctions of the problem

Soln $p(x) = x, \quad q(x) = 0, \quad r(x) = 1/x,$

$a_1 = 0, \quad a_2 = 1, \quad b_1 = 1, \quad b_2 = 0, \quad a = 1, \quad b = e.$

$\therefore (xy')' + \lambda/x \, y = 0, \quad y'(1) = 0, \quad y(e) = 0$

$xy'' + y' + \lambda/x \, y = 0 \Rightarrow x^2 y'' + xy' + \lambda y = 0$

Let $x = e^t$

$\Rightarrow \frac{d}{dx} = \frac{d}{dt}\cdot\frac{dt}{dx} = \frac{d}{dt}\Big/\frac{dx}{dt} = \boxed{e^{-t}\frac{d}{dt}}.$

Let $x = e^t$
$y' = \frac{d}{dx} = \frac{d}{dt}\cdot\frac{dt}{dx} = e^{-t}\frac{d}{dt}$
$y'' = \frac{d^2}{dx^2} = \frac{d}{dx}\cdot\frac{d}{dx} = e^{-2t}\left[\frac{d^2}{dt^2} - \frac{d}{dt}\right]$

→ Similarly $\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = e^{-t}\frac{d}{dt}\left(e^{-t}\frac{d}{dt}\right)$

$$= \boxed{e^{-2t}\left\{\frac{d^2}{dt^2} - \frac{d}{dt}\right\}}$$

Substituting into the general Equation we have

$$e^{2t}\cdot e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + e^{t}\cdot e^{-t}\frac{dy}{dt} + \lambda y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + \lambda y = 0 \iff \frac{d^2y}{dt^2} + \lambda y = 0.$$

$y'(1) = 0,\quad x = 1 \Rightarrow e^t = 1 \Rightarrow t = 0 \quad \therefore y'(0) = 0$

$y(e) = 0,\quad x = e \Rightarrow e^t = e \Rightarrow t = 1 \quad \therefore y(1) = 0$ $\quad 0 \le t \le 1$

The auxiliary Eqn is

$m^2 + \lambda = 0 \quad \therefore m = \pm\sqrt{\lambda}\, i$

$y(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t$

$\therefore y'(t) = -A\sqrt{\lambda}\sin\sqrt{\lambda}t + B\sqrt{\lambda}\cos\sqrt{\lambda}t$

$\therefore y'(0) = 0 \Rightarrow -A\sqrt{\lambda}\sin\sqrt{\lambda}\cdot 0 + B\sqrt{\lambda}\cos\sqrt{\lambda}\cdot 0 = 0$

$\Rightarrow B\sqrt{\lambda}\cdot 1 = 0 \Rightarrow B = 0$

$y(t) = A\cos\sqrt{\lambda}t$

$y(1) = 0 \Rightarrow A\cos\sqrt{\lambda}\cdot 1 = 0$

$\therefore A\cos\sqrt{\lambda} = 0 \Rightarrow \cos\sqrt{\lambda} = 0$

$\therefore \sqrt{\lambda} = \frac{n\pi}{2},\ n = 1, 3, 5 \Rightarrow \lambda = \frac{n^2\pi^2}{4}$

$\therefore$ The set of eigen values $\lambda_n = \left\{\frac{n^2\pi^2}{4}\right\} = \{n = 1, 3, 5 \ldots$

$\lambda_n = \left\{\frac{\pi^2}{4}, \frac{9\pi^2}{4}, \frac{25\pi^2}{4} \ldots\right.$

$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

To show that the set is orthogonal:

The corresponding eigenfunctions are

$y_n(t) = A_n \cos \frac{n\pi}{2} t$. $\quad r(x) = 1/x$

$$\int_0^1 y_n(t)\, y_m(t)\, dt = \int_0^1 A_n \cos \frac{n\pi}{2} t + A_m \cos \frac{n\pi t}{2}\, dt$$

$$\Rightarrow A_n A_m \int_0^1 \cos \frac{n\pi}{2} t + \cos \frac{n\pi t}{2}\, dt$$

$$= \frac{A_n A_m}{2} \int_0^1 \cos(n-m)\frac{\pi t}{2} + \cos(n+m)\frac{\pi}{2} t\, dt$$

$$= \frac{A_n A_m}{2} \left[\frac{2}{(n-m)\pi} \sin (n-m)\frac{\pi t}{2} \Big|_0^1 + \frac{2 \sin (n+m)\frac{\pi t}{2}}{(n+m)\pi} \Big|_0^1 = 0\right.$$

$$\int_0^1 y_n(t)^2\, dt \Longrightarrow \int_0^1 A_n^2 \cos^2 \frac{n\pi}{2}\, dt = 1$$

$$I = \int_0^1 \frac{A_n^2}{2} (\cos n\pi t + 1)\, dt = 1$$

$$= \frac{A_n^2}{2} \left[\frac{\sin n\pi t}{n\pi} + t\right]_0^1 = 1$$

$$\Rightarrow \frac{A_n^2}{2} [0 + 1] = 1 \quad \therefore A_n^2 = 2 \quad \therefore A_n = \sqrt{2}$$

$$y_n(t) = \sqrt{2} \cos \frac{n\pi}{2} t, \quad n = 1, 3, 5 \ldots$$

but $t = \ln x$, $y_n(x) = \sqrt{2} \cos \left(\frac{n\pi \ln x}{2}\right)$, $n = 1, 3 \ldots$

$$y_n(x) = \left\{\sqrt{2} \cos \left(\frac{2n+1}{2}\right)\pi \ln x, \quad n = 0, 1, 2 \ldots\right.$$

EX Show that the equation: $(xy')' + \left(\frac{\lambda+1}{x}\right) y = 0$; $y'(1) = 0$, $y'(e) = 0$ is a S-L Equation. Determine the eigenvalue and the corresponding eigenfunctions of the problem.

Show that ~~the~~ eigenfunctions form an orthogonal and orthonormal set.

~~Assignment~~

1- Use Gamma & Beta function to Evaluate

i- $\int_0^1 x^5 (\ln x)^3 \, dx$ ii- $\int_0^{1/3} x^2 \sqrt{1-9x^2} \, dx$

2- Show that $(xy')' + \frac{(\lambda+1)y}{x} = 0$, $y(1) = 0$, $y(e) = 0$ is a Sturm-Liouville equation. Solve to determine the ~~singularity~~ Eigenvalues and the corresponding eigenfunctions. Hence show that the Eigenfunction form an orthogonal and orthonormal set.

Some Important fxs & Derivatives

$\int \cos x = \sin x$

$\int \sin x = -\cos x$

$\sin^{-1}(0) = n\pi$

$\cos^{-1}(0) = \frac{n\pi}{2}$

$\sin^2 A = \frac{1}{2}[1 - \cos 2A]$

$\cos^2 A = \frac{1}{2}[1 + \sin 2A]$

$\sin n \cdot \sin m = \frac{1}{2}[\cos(n-m) - \cos(n+m)]$

$\cos n \cdot \cos m = \frac{1}{2}[\cos(n+m) + \cos(n-m)]$

ordinary pt

$y(x) = \sum_{n=0}^{\infty} a_n x^n$

singular pt

$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}$

ss - - +

cc + + -

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3. SERIES SOLUTION OF O.D.E.:

Consider the second order ordinary differential equation of the form:

$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad a_0 \neq 0$ — (1)

Used $a_0(x) \neq 0$ doesn't vanish, singu. $a_0(x) = 0$ vanish

The solution of equation (1) can be expressed as a series about the point $x = a$

$y(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$

Regular or Ordinary point: A point $x = a$ is called an ordinary point of the differential equation (1) if $a_0(a) \neq 0$. Otherwise the point is a singular point. Since $a_0(0) \neq 0$.

Examples

i. $(1+x^2)y'' - xy' + 2y = 0$, the point $x = 0$ is an ordinary point, $a_0 = 1+x^2$, $a_1 = -x$, $a_2 = 2$, $y(0)=1$, $y'(0)=2$

ii. $2x^2y'' + xy' - (1+x^2)y = 0$, $x = 0$ is a singular point as since $a_0(x) = 2x^2$ is zero at $x=0$

Soln

If $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 \dots a_nx^n \dots$

$y'(x) = a_1 + 2a_2x + 3a_3x^2 \dots + na_nx^{n-1} + \dots$

$y''(x) = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots$

Substituting into the given equation we have

$(1+x^2)y'' - xy' + 2y = 0$

$2a_2 + 6a_3x + \cancel{\ } \ldots\ldots - - - n(n-1)a_n x^{n-2} + \ldots$

$+ 2a_2x^2 + 6a_3x^3 + \ldots\ldots n(n-1)a_n x^n + \ldots$

$- a_1x - 2a_2x^2 - 3a_3x^3 + \ldots\ldots - n a_n x^n \ldots$

$+ 2a_0 + 2a_1 x + 2a_2x^2 + 3a_3x^3 + \ldots + 2a_nx^n = 0$

Comparing ~~form~~ :

$2a_2 + 2a_0 = 0 \therefore \boxed{a_2 = -a_0}$

$6a_3 - a_1 + 2a_1 = 0 \Rightarrow 6a_3 + a_1 = 0 \therefore \boxed{a_3 = \frac{-a_1}{6}}$

$(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 2a_n = 0$

$(n+2)(n+1)a_{n+2} + n^2a_n - na_n - na_n + 2a_n = 0$

$(n+2)(n+1)a_{n+2} + (n^2 - 2n + 2)a_n = 0$

$a_{n+2} = \frac{-(n^2-2n+2)a_n}{(n+2)(n+1)}$. (Recursive formula RF)

$a_2 = -\frac{2a_0}{2} = -a_0$ //

$a_3 = -\frac{(1-2+2)a_1}{3\cdot 2} = \frac{-a_1}{6}$ //

$a_4 = -\frac{a_2}{6} = -\frac{(-a_0)}{6} = \frac{a_0}{6}$ //

$a_5 = \frac{-(9-6+2)a_3}{5\cdot 4} = \frac{-5a_3}{5\cdot 4} = -\frac{a_3}{4} = -\frac{1}{4}\left(\frac{-a_1}{6}\right) = \frac{a_1}{24}$ //

substituting into the series given :

$\therefore y(x) = a_0 + a_1x - a_0x^2 - \frac{a_1}{6}x^3 + \frac{a_0}{6}x^4 + \frac{a_1}{24}x^5$

$\therefore y(x) = a_0\left(1 - x^2 + \frac{x^4}{6} + \ldots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^5}{24} - \ldots\right)$

$y(x) = 1 \Rightarrow y(x) = a_0 + a_1 x + a_2 x^2 \ldots$

$y(0) = 1 \Rightarrow a_0 = 1$

$y'(0) = 2 \Rightarrow y'(x) = a_1 + 2a_2 x + \ldots$

$y(0) = 2 \Rightarrow a_1 = 2$

$\therefore y(x) = 1 - x^2 + \frac{x^4}{6} + \ldots + 2\left(x - \frac{x^3}{6} + \frac{x^5}{24} + \ldots\right)$

$\therefore y(x) = 1 + 2x - x^2 - \frac{x^3}{3} + \frac{x^4}{5} + \frac{x^5}{12} + \ldots$

Ex Use the series solution method to solve:

i) $y'' - 3xy' + 5y = 0$, $y(0) = 1$, $y'(0) = -1$

ii) $2y'' - xy' + y = 1 + x - x^2$, $y(0) = 1$, $y'(0) = 0$

iii) $(1+x^2)y'' + xy' - y = 0$, $y(1) = 2$, $y'(1) = 0$ about $x = 1$

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Series solution about a singular point at $x = 0$

Frobenius method

* Regular singular point:

A point $x = a$ is said to be a regular singular point of the differential equation.

$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ if the functions: $\frac{a_1(x)(x-a)}{a_0(x)}$ and $\frac{a_2(x)(x-a)^2}{a_0(x)}$ are analytic about $x = a$ (i.e. can expanded as a power series about $x = a$)

Otherwise the point $x = a$ is an Irregular singular point.

Example:

1) $2x^2y'' - xy' + (x^2+1)y = 0$ is $x=0$ a regular or an irregular singular point?

Solution

$a_0(x) = 2x^2$, $a_1 = -x$, $a_2 = x^2+1$

$$p(x) = \frac{a_1(x)(x-a)}{a_0(x)}, \quad Q(x) = \frac{a_2(x)(x-a)^2}{a_0(x)}$$

$$\therefore p(x) = \frac{-x\cdot(x-0)}{2x^2} = \frac{-x^2}{2x^2} = -1/2$$

$$Q(x) = \frac{(x^2+1)(x-0)^2}{2x^2} = \frac{\cancel{x^2}(1+x^2)}{2\cancel{x^2}} = 1/2\,(1+x^2)$$

Clearly $p(x) = -1/2$ is continuous at $x=0$ and $Q(x) = \frac{1}{2}(1+x^2)$ is continuous at $x=0$. Thus $p(x)$ and $Q(x)$ are analytic at $x=0$. Hence $x=0$ is a regular singular point.

2) $x^3(x-2)^2y'' + 5(x+2)(x-2)y' + 3x^2y = 0$

Solution

$a_0(x) = x^3(x-2)^2$, $a_1(x) = 5(x+2)(x-2)$, $a_2(x) = 3x^2$

$x^3(x-2)^2 = 0 \Rightarrow x=0$ and $x=2$ are singular points

$$p(x) = \frac{a_1(x)(x-a)}{a_0(x)} = \frac{5(x+2)\cancel{(x-2)}(x-0)}{x^3(x-2)^{\cancel{2}}} = \frac{5x(x+2)}{x^3(x-2)}$$

$$\Rightarrow p(x) = \frac{5(x+2)}{x^2(x-2)}$$ $\therefore p(x)$ is not continuous at $x=0$.

Thus $x=0$ is an irregular singular point.

Similarly $$Q(x) = \frac{a_2(x)(x-a)}{a_0(x)}$$

$$Q(x) = \frac{3x^2(x-0)^2}{x^3(x-2)^2} = \frac{3x^4}{x^3(x-2)^2} = \frac{3x}{x^2-4x+4}$$

$Q(x)$ is not continuous at $x=0$

$\therefore Q(x)$ is not analytic at $x=0$

If $x=2$, then

$$p(x) = \frac{a_1(x)(x-a)}{a_0(x)} = \frac{5(x+2)\cancel{(x-2)}(x-2)}{x^3\cancel{(x-2)^2}} = \frac{5(x+2)}{x^3}$$

$$\therefore \lim_{x\to 2} p(x) = \frac{5(x+2)}{x^3} = \frac{5(4)}{2^3} = \frac{20}{8} = 5/2$$

$p(x)$ is continuous at $x=2$

$$Q(x) = \frac{a_2(x)(x-a)}{a_0(x)} = \frac{3x^2(x-2)^2}{x^3(x-2)^2} = \frac{3}{x}$$

$$\lim_{x\to 2} Q(x) = \frac{3}{x} = 3/2$$

$Q(x)$ is continuous at $x=2$

Thus $Q(x)$ is a regular singular point at $x=2$.

NB: If $x=a$ is a regular singular point to the differential Equation:

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

then the solution to the differential Equation can be expressed as series of the form:

$$y(x) = x^m \sum_{n=0}^{\infty} a_n (x-a)^n,$$ where

m is a number to be determined.

$$y(x) = x^m [a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n \ldots] \text{ or}$$

$$y(x) = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} \ldots + a_n x^{m+n} + \ldots$$

Substituting the above series into the given differential equation leads to the Indicial Equation.

Examples

1. find the series solution of the differential equation $2x^2y'' - xy' + (x^2+1)y = 0$ about $x=0$.

solution

Since $x=0$ is a singular point.

let $y = x^m \sum_{n=0}^{\infty} a_n x^n$

$$y(x) = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} \ldots + a_n x^{m+n} + \ldots$$

$$y'(x) = m a_0 x^{m-1} + (m+1)a_1 x^m + (m+2)a_2 x^{m+1} + (m+3)a_3 x^{m+2} \ldots + (m+n)a_n x^{m+n-1} + \ldots$$

$$y''(x) = m(m-1)a_0 x^{m-2} + m(m+1)a_1 x^{m-1} + (m+1)(m+2)a_2 x^m + (m+2)(m+3)a_3 x^{m+1} + \ldots$$
$$(m+n-1)(m+n)a_n x^{m+n-2} + \ldots$$

∴ Substituting into the given equation yields:

$2m(m-1)a_0x^m + 2m(m+1)a_1x^{m+1} + 2(m+1)(m+2)a_2x^{m+2}$
$+ 2(m+2)(m+3)a_3x^{m+3} + \dots + 2(m+n-1)(m+n)a_nx^{m+n} \dots$
$- ma_0x^m - (m+1)a_1x^{m+1} - (m+2)a_2x^{m+2} + (m+3)a_3x^{m+3} + \dots$
$\dots - (m+n)a_nx^{m+n} + \dots$
$+ a_0x^{m+2} + a_1x^{m+3} + a_2x^{m+4} + a_3x^{m+5} + \dots a_nx^{m+n+2} + \dots$
$+ a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} \dots + a_nx^{m+n} + \dots = 0$

∴ The indicial equation is obtained by equating the coefficients of the least power of x to zero (in the equation).

$2m(m-1)a_0 - ma_0 + a_0 = 0$

$(2m^2 - 2m - m + 1)a_0 = 0 \Rightarrow 2m^2 - 3m + 1 = 0$

$2m^2 - 2m - m + 1 = 0$

$\therefore 2m(m-1) - 1(m-1) = 0$

$\therefore (2m-1)(m-1) = 0 \quad \therefore m = 1/2, \ m = 1$

To determine the recursive formula:

$2(m+n)(m+n-1)a_n - (m+n)a_n + a_{n-2} + a_n = 0$

$(m+n)(2m+2n-3)a_n + a_{n-2} + a_n = 0$

$[(m+n)(2m+2n-3)+1]a_n + a_{n-2} = 0$

$\therefore a_n = -\dfrac{a_{n-2}}{[(m+n)(2m+2n-3)+1]} \qquad n \geq 2$

$m = 1/2: \ a_n = \dfrac{-a_{n-2}}{[(n+1/2)(2\cdot 1/2+2n-3)+1]} = \dfrac{-a_{n-2}}{(2n+1)(n-1)+1}$

$\therefore \quad a_n = \dfrac{-a_{n-2}}{2n^2-n}, \quad n \geq 2$

$a_2 = \dfrac{-a_0}{6}, \quad a_3 = \dfrac{-a_1}{15}, \quad a_4 = \dfrac{-a_2}{28} = \dfrac{a_0}{28 \cdot 6} = \dfrac{a_0}{168}$

$a_5 = \dfrac{-a_3}{45} = \dfrac{a_1}{15 \cdot 45} = \dfrac{a_1}{675}$

$\therefore y_1(x) = x^{1/2}\left[a_0 + a_1 x - \dfrac{a_0}{6}x^2 - \dfrac{a_1}{15}x^3 + \dfrac{a_0}{168}x^4 + \dfrac{a_1 x^5}{675} + \cdots\right.$

<u>Check $m=1$</u>

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After strike

Recall: If $a_0(x) = 0$ is an ordinary or Reguler

If $a_0(x) \neq 0$ it is an irregular

Singular point < Reguler singular points / Irregular singular point.

(a) NB: A singular point is said to be reguler singular point if the functions $\dfrac{(x-x_0)\,a_1(x)}{a_0(x)}$ and $\dfrac{(x-x_0)^2\,a_2(x)}{a_0(x)}$ are analytic function at $x = x_0$

(b) A singular point is said to be irregular singular point if one of the functions is a non-analytic function at $x = x_0$.

Example: Given the differential Equation

$2x^2y'' - xy' + (1-x^2)y = 0$

soln

$a_0(x) = 2x^2, \quad a_1(x) = -x, \quad a_2(x) = 1-x^2$

from $a_0(x) = 2x^2$

$\therefore 2x^2 = 0 \quad \therefore x = 0$

$\Rightarrow x = 0$ is a singular point

$x=0$ is a regular singular point if the function $(x-x_0)\frac{a_1(x)}{a_0 x}$ is analytic at $x=0$

$$\therefore (x-0)\frac{(-x)}{2x^2} = \frac{(x)(-x)}{2x^2} = \frac{-\cancel{x^2}}{2\cancel{x^2}} = -1/2$$

and
$$(x-x_0)^2\frac{a_2(x)}{a_0(x)} = \frac{(x-0)^2(1-x^2)}{2x^2} = \frac{\cancel{x^2}(1-x^2)}{2\cancel{x^2}} = \frac{1}{2}(1-x^2)$$

Since $-1/2$, $1/2(1-x^2)$ are analytic at $x=0$, then $x=0$, is a Regular singular point.

Exercise

$(1-x^2)y'' + xy' + x^2y = 0$. Given the equation show that $x=\pm1$ are singular points. Hence discriminate between them.

If $x = x_0$ is an ordinary point then the solution of the equation can be expressed in the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

Example

Express the solution of the differential equation $(1+x^2)y'' - 3xy' - 5y = 0$, $y(0) = 1$, $y'(0) = 0$ on a series solution of $(x-0)$; since $x=0$ is ordinary point.

~~Soln~~

$$(1+x^2)y'' - 3xy' - 5y = 0$$

let $\cancel{y(x) = \sum_{n=0}^{\infty} a_n (x-a}$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = n \sum_{n=0}^{\infty} a_n x^{n-1}$$

$$y''(x) = n(n-1) \sum_{n=0}^{\infty} a_n x^{n-2}$$

$$n(n-1)(1+x^2) \sum_{n=0}^{\infty} a_n x^{n-2} - 3nx \sum_{n=0}^{\infty} a_n x^{n-1} - 5 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$n(n-1) \sum_{n=0}^{\infty} a_n x^{n-2} + n(n-1) \sum_{n=0}^{\infty} a_n x^n - 3n \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 5a_n x^n = 0$$

Introducing +2 to remove -2 so that everything will be in terms of x^n

$$(n+2)(n+1) \sum_{n=0}^{\infty} a_{n+2} x^n + n(n-1) \sum_{n=0}^{\infty} a_n x^n - 3n \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 5a_n x^n = 0$$

$$\left[(n+2)(n+1) a_{n+2} + n(n-1) a_n - 3n a_n - 5a_n\right] \sum_{n=0}^{\infty} x^n = 0$$

$$\left[(n+2)(n+1)a_{n+2} + (n^2-n)a_n - 3n\,a_n - 5a_n\right] = 0$$

$$\left[(n+2)(n+1)a_{n+2} + (n^2-n-3n-5)a_n\right] = 0$$

$$(n+2)(n+1)a_{n+2} + (n^2-4n-5)a_n = 0$$

$$\therefore a_{n+2} = \frac{-(n^2-4n-5)}{(n+2)(n+1)}$$

$$= \frac{-\left[(n-5)\cancel{(n+1)}\right]a_n}{(n+2)\cancel{(n+1)}}$$

$$= \frac{-(n-5)a_n}{(n+2)} \Rightarrow \frac{(5-n)a_n}{(n+2)}$$

$$\therefore \boxed{a_{n+2} = \frac{(5-n)a_n}{(n+2)}} \text{ Recursive formula}$$

i.e $a_2 = \frac{5a_0}{2}$ for $n=0$

$a_3 = \frac{4a_1}{3}$ for $n=1$

$a_4 = \frac{3a_2}{4} = \frac{15a_0}{8}$ for $n=2$

$a_5 = \frac{2a_3}{5} = \frac{8a_1}{15}$ for $n=3$

$$\therefore y = \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \text{-----}$$

$$\therefore y(x) = a_0 + a_1 x + \frac{5a_0}{2}x^2 + \frac{4a_1}{3}x^3 + \frac{15a_0}{8}x^4 + \frac{8a_1}{15}x^5 \text{-----}$$

$\therefore y(x) = a_0\left(1 + \frac{5}{2}x^2 + \frac{15}{8}x^4 + \cdots\right)$
$\quad + a_1\left(x + \frac{4}{3}x^3 + \frac{8}{15}x^5 + \cdots\right)$

$\because y(0) = 1 \Rightarrow 1 = a_0 \quad \therefore a_0 = 1$

$y'(0) = 0 \Rightarrow 0 = a_1 \quad \therefore a_1 = 0$

$\therefore y = 1 + \frac{5}{2}x^2 + \frac{15x^4}{8} + \cdots$

If $x = x_0$ is a singuler point of The diff $a_0(x)y'' + a_1(x)y' + a_2(x)$ then

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^{m+n}}$$

Example

$$2x^2y'' - xy' + (1 - x^2)y = 0$$

Since $x = 0$ is a singuler point then

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

Soln

$$y = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$y' = \sum_{n=0}^{\infty} a_n x^{m+n} = (m+n)\sum_{n=0}^{\infty} a_n x^{m+n-1}$$

$$y'' = (m+n)\sum_{n=0}^{\infty} a_n x^{m+n-1} = (m+n-1)(m+n)\sum_{n=0}^{\infty} a_n x^{m+n-2}$$

$$\therefore 2x^2(m+n-1)(m+n)\sum_{n=0}^{\infty} a_n x^{m+n-2} - x(m+n)\sum_{n=0}^{\infty} a_n x^{m+n-1} + (1-x^2)\sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$2(m+n)(m+n-1)\sum_{n=0}^{\infty} a_n x^{m+n} - (m+n)\sum_{n=0}^{\infty} a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} - \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$$

$$2(m+n)(m+n-1)\sum_{n=0}^{\infty} a_n x^{m+n} - (m+n)\sum_{n=0}^{\infty} a_n x^{m+n} + \sum_{n=0}^{\infty} a_n x^{m+n} - \sum_{n=0}^{\infty} a_{n-2} x^{m+n} = 0$$

$$\left[2(m+n)(m+n-1)a_n - (m+n)a_n + a_n\right]\sum_{n=0}^{\infty} x^{m+n} - a_{n-2}\sum_{n=0} x^{m+n} = 0$$

$$\Rightarrow \left[2(m+n)(m+n-1) - (m+n)a_n + a_n\right]\Sigma x^{m+n} - a_{n-2}\Sigma x^{m+n} = 0$$

$$\left[(m+n)(2m+2n-3)+1\right]a_n = a_{n-2}$$

$$a_n = \frac{a_{n-2}}{(m+n)(2m-2n-3)+1}$$

for Indicial equation:

$$\left[2m(m-1) - m + 1\right]a = 0$$

$$2m^2 - 3m + 1 = 0$$

$$2m^2 - 2m - m + 1 = 0$$

$$2m(m-1) - 1(m-1) = 0$$

$$\therefore (2m-1)(m-1) = 0$$

$$m = 1, \tfrac{1}{2}$$

similarly the recursive formula is given by

$$2(m+n)(m+n-1)a_n - (m+n)a_n + a_n - a_{n-2} = 0$$

$$(m+n)(2m+2n-2)a_n + a_n = a_{n-2}$$

$$\left[a_n = \frac{a_{n-2}}{(m+n)(2m+2n-3)+1}\right] n \geq 2$$ — Recursive formula

~~for m=1~~

~~$a_n = a_{n-2}$~~

for $m=1$

$$a_n = \frac{a_{n-2}}{(n+1)(2n-1)+1} = \frac{a_{n-2}}{2n^2+n} = a_2 = \frac{a_0}{10},$$

$$a_3 = \frac{a_1}{21}, \quad a_4 = \frac{a_2}{36} = \frac{a_0}{360}, \quad a_5 = \frac{a_3}{55} = \frac{a_1}{1155}$$

$$y(x) = x^m(a_n x^n) = y(x) = a_n x^{m+n}$$

$$\Rightarrow y_1(x) = x^1\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots\right]$$

$$\Rightarrow y_1(x) = x\left(a_0 + a_1 x + \frac{a_0}{10}x^2 + \frac{a_1}{21}x^3 + \frac{a_0 x^4}{360} + \frac{a_1 x^5}{1155} + \cdots\right.$$

$$y_1(x) = a_0\left(x + \frac{x^3}{10} + \frac{x^5}{360} + \cdots\right) + a_1\left(x^2 + \frac{x^4}{21} + \frac{x^6}{1155} \cdots\right)$$

for $m = 1/2$

$$a_n = \frac{a_{n-2}}{(n+1/2)(2n-2)+1} = \frac{a_{n-2}}{(2n+1)(n-1)+1} = \frac{a_{n-2}}{2n^2-n} = a$$

$$a_2 = \frac{a_0}{6}, \quad a_3 = \frac{a_1}{15}, \quad a_4 = \frac{a_2}{28} = \frac{a_0}{168}, \quad a_5 = \frac{a_3}{45} = \frac{a_1}{675}$$

$$y(x) = x^{1/2}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots\right]$$

$$y(x) = x^{1/2}\left[a_0\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + \cdots\right) + a_1\left(x + \frac{x^3}{15} + \frac{x^5}{675} + \cdots\right)\right]$$

$$y_2(x) = a_0\left(x^{1/2} + \frac{x^{5/2}}{6} + \frac{x^{9/2}}{168} + \cdots\right) + a_1\left(x^{3/2} + \frac{x^{7/2}}{15} + \frac{x^{11/2}}{675} + \cdots\right)$$

Note that:

1. If the roots of the Indicial Equation differ by a number (not an Integer); then the solution of the equation a given by $y(x) = c_1 y\big|_{m=m_1} + c_2 y\big|_{m=m_2}$

2) If the roots differ by an integer, then the solution can be expressed as

$$y(x) = C_1 y\Big|_{m=m_1} + C_2 \frac{\partial y}{\partial m}\Big|_{m=m_2}$$

3) If the roots are repeated i.e $m=m_1 \cdot m=m_1$
then $y(x) = C_1 y_{m=m_1} + C_2 \frac{\partial y}{\partial m}\Big|_{m=m_1}$

Exercise

Use the fröbenius method solve the differential equ.

i) $2xy'' + y' + xy = 0$

ii) $2x^2y'' ~~+~~ xy' + (x^2+1)y = 0$

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* Some special ordinary Differential Equations

① Bessel's differential Equations: This Equation has the general form

$$x^2y'' + xy' + (x^2-k^2)y = 0,$$

Where k is any positive or negative number.

~~differential equation~~

The solution of the Bessel's differential equation yields polynomials functions called Bessel's functions (of the first and second kind).

let $y = \sum_{n=0}^{\infty} a_n x^{m+n} \Rightarrow y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_nx^{m+n}$

$y' = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} \dots + (m+n)a_nx^{m+n-1} \dots$

$y'' = m(m-1)a_0x^{m-2} + m(m+1)a_1x^{m-1} + (m+1)(m+2)a_2x^m + \dots + (m+n-1)(m+n)a_nx^{m+n-2} + \dots$

substituting into the Bessel's diff. equn gives

$$\Rightarrow m(m-1)a_0x^m + m(m+1)a_1x^{m+1} + (m+1)(m+2)a_2x^{m+2} + \cdots (m+n)(m+n-1)a_nx^{m+n} \cdots$$
$$+ ma_0x^m + (m+1)a_1x^{m+1} + (m+2)a_2x^{m+2} + \cdots (m+n)a_nx^{m+n} + \cdots$$
$$+ a_0x^{m+2} + a_1x^{m+3} + a_2x^{m+4} + \cdots a_nx^{m+n+2}$$
$$- k^2a_0x^m - k^2a_1x^{m+1} - k^2a_2x^{m+2} + \cdots - k^2a_nx^{m+n} \cdots = 0$$

The indicial equation is given by least power of x

$$m(m-1)a_0 + ma_0 - k^2a_0 = 0$$
$$\Rightarrow m^2a_0 - ma_0 + ma_0 - k^2a_0 = 0$$
$$\Rightarrow (m^2 - k^2)a_0 = 0$$
$$\therefore m^2 - k^2 = 0 \text{ or } \boxed{m = \pm k}$$

Similarly the recursive formula is:

$$(m+n)(m+n-1)a_n + (m+n)a_n + a_{n-2} - k^2a_n = 0$$
$$[(m+n)(m+n-1) + 1) - k^2]a_n = -a_{n-2}$$
$$\Rightarrow a_n = \frac{-a_{n-2}}{(m+n)(m+n) - k^2} \quad n \geq 2$$

Case 1 : $m = k$; $a_n = \frac{-a_{n-2}}{(k+n)^2 - k^2} = \frac{-a_{n-2}}{2nk + n^2}$

$$a_2 = \frac{-a_0}{4k+4} = -\frac{a_0}{4(k+1)} \qquad a_4 = -\frac{a_2}{8k+16} = \frac{-a_2}{8(k+2)} = \frac{a_0}{4 \cdot 8(k+1)(k+2)}$$

$$a_6 = \frac{-a_4}{12k+36} = \frac{-a_4}{12(k+3)} = \frac{-a_0}{4 \cdot 8 \cdot 12(k+1)(k+2)(k+3)}$$

$$y_k(x) = x^k\left[a_0 - \frac{a_0x^2}{4(k+1)} + \frac{a_0x^4}{4 \cdot 8(k+1)(k+2)} - \frac{a_0x^6}{4 \cdot 8 \cdot 12(k+1)(k+2)(k+3)} + \cdots\right]$$

$$y_k(x) = a_0 x^k \left[1 - \frac{x^2}{4(k+1)} + \frac{x^4}{4\cdot 8(k+1)(k+2)} - \frac{x^6}{4\cdot 8 \cdot 12(k+1)(k+2)(k+3)} + \cdots \right.$$

$$y_k(x) = a_0 x^k \left[1 - \frac{x^2}{2^2(k+1)!} + \frac{x^4}{2^4 \cdot 2!(k+1)(k+2)} - \frac{x^6}{2^6 \cdot 3!(k+1)(k+2)(k+3)} + \cdots \right.$$

Taket $a_0 = \frac{1}{2^k k!}$

$$y_k(x) = x^k \left[\frac{1}{2^k k!} - \frac{x^2}{2^{k+2} 1!(k+1)!} + \frac{x^4}{2^{k+4} 2!(k+2)!} - \frac{x^6}{2^{k+6} 3!(k+3)!} + \cdots \right.$$

$$y_k(x) = x^k \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{k+2n} \cdot n!(k+n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{k+2n}}{2^{k+2n} n!(k+n)!}$$

$$\boxed{J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{k+2n}}{2^{k+2n} n!(k+n)!}}$$

Generalised form

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{k+2n}}{2^{k+2n} \Gamma(n+1)\Gamma(k+n+1)}$$

Case 2: $m = -k \Rightarrow a_n = \frac{-a_{n-2}}{(n-k)^2 - k^2} = \frac{-a_{n-2}}{n^2 - 2nk}$

$$a_2 = \frac{-a_0}{4-4k} = \frac{-a_0}{4(1-k)}, \quad a_4 = \frac{-a_2}{16-8k} = \frac{-a_2}{8(2-k)} = \frac{a_0}{4\cdot 8(1-k)(2-k)}$$

$$a_6 = \frac{-a_4}{36-12k} = \frac{-a_4}{12(3-k)} = \frac{-a_0}{4\cdot 8\cdot 12(1-k)(2-k)(3-k)}$$

Conforming to the argument we have

$$\boxed{J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-k}}{2^{2n-k} \cdot n!(n-k)!}}$$

$$y(x) = A\,p(x) + B\,q(x) = A\,J_k(x) + B\,J_k(x)$$

where $p(x)$ and $q(x)$ are the bessel's functions of the differential equation of the first kind & second kind respectively

To solve the differential equ:

$$x^2y'' + xy' + (x^2-1)y = 0$$

$\Rightarrow k^2 = 1 \quad \therefore k = \pm 1$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}\cdot n!(n+1)!} = \frac{x}{2} - \frac{x^3}{8\cdot1\cdot2!} + \frac{x^5}{32\cdot2!\,3!} - - - -$$

$$\therefore J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - - - - -$$

If $J_k(x)$ is the bessel's function show that

i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Soln

from $$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{k+2n}}{2^{k+2n}\cdot n!(k+n)!}$$

$$\therefore J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{1/2+2n}}{2^{1/2+2n}\cdot n!(n+1/2)!}$$

$$\therefore J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}!} - \frac{x^2}{4\cdot\frac{3}{2}!} + \frac{x^4}{16\cdot2!\,\frac{5}{2}!} - - - -\right]$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma 3/2} - \frac{x^2}{4\cdot\Gamma 5/2} + \frac{x^4}{16\cdot2\cdot\Gamma 7/2} - - - -\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} - \frac{x^2}{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} + \frac{x^4}{16\cdot2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} - - - -\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\cdot\frac{1}{\frac{1}{2}\cdot\sqrt{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - - - -\right]$$

$$J_k(x) = \frac{x^k}{2^k\Gamma(k+1)}\left[1 - \frac{x^2}{2\cdot2(k+1)} + \frac{x^4}{2\cdot4\cdot2^2(k+1)(k+2)} - - -\right]$$

$$\therefore J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}\,x}\cdot\frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right.$$

$$= \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left(\sin x\right)$$

$$\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x}$$

BA: 9/2/23

BESSEL'S FUNCTIONS

NOTE THAT: If $J_K(x)$ is the bessel's function of the first kind show that

i) $\frac{d}{dx}\left(x^K J_K(x)\right) = x^K J_{K-1}(x)$

ii) $\frac{d}{dx}\left(x^{-K} J_K(x)\right) = -x^{-K} J_{K+1}(x)$

iii) $\frac{d}{dx}\left(J_K(x)\right) = \frac{1}{2}\left[J_{K-1}(x) - J_{K+1}(x)\right]$

iv) $J_{K+1}(x) + J_{K-1}(x) = \frac{2K}{x} J_K(x)$. Hence if $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$, find $J_{3/2}(x)$.

If $J_K(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{K+2n}}{2^{K+2n}\cdot n!(K+n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{K+2n}}{2^{K+2n}\cdot\Gamma(n+1)\Gamma(K+n+1)}$

$$\therefore \frac{d}{dx}\left[x^K J_K(x)\right] = \frac{d}{dx}\left[x^K \sum_{n=0}^{\infty} \frac{(-1)^n x^{K+2n}}{2^{K+2n}\cdot n!(K+n)!}\right]$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2K+2n}}{2^{K+2n}\cdot n!(K+n)!}\right]$$

$(n-1)!$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2k+2n)\, x^{2k+2n-1}}{2^{k+2n}\, n!\,(k+n)!} = 2(k+n) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2k+2n-1}}{2^{k+2n} \cdot n!\,(k+n)!}$$

$$= \cancel{2}(k+n) \cdot x^k \sum_{n=0}^{\infty} \frac{(-1)^n x^{k+2n-1}}{2^{k+2n-1} \cdot n!\,\cancel{(k+n)}(k+n-1)!}$$

$$= x^k \sum_{n=0}^{\infty} \frac{(-1)^n x^{k+2n-1}}{2^{k+2n-1} \cdot n!\,(k+n-1)!} = x^k J_{k-1}(x)$$

ii) $$\frac{d}{dx}\left[x^{-k} J_k(x)\right] = \frac{d}{dx}\left[x^{-k} \sum_{n=0}^{\infty} \frac{(-1)^n x^{k+2n}}{2^{k+2n} \cdot n!\,(k+n)!}\right]$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{k+2n} \cdot n!\,(k+n)!}\right] = 2n \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{k+2n} \cdot n!\,(k+n)!}$$

$=$

$$2n \sum \frac{(-1)^n x^{2n-1}}{2^{k+2n}\, n!\,(k+n)!} = 2n \sum \frac{(-1)^n x^{2n-1}}{2^{k+2n-1}\,(n-1)!\,(k+n-1)!} =$$

iii) $\frac{d}{dx} J_k(x) = J_k'(x) = \frac{1}{2}\left[J_{k-1}(x) - J_{k+1}(x)\right]$

~~proof:~~

~~from (1) we have~~ $\frac{d}{dx}\left[x^k J_k(x)\right] = x^k J_{k-1}(x)$

$\Rightarrow x^k J_k'(x) + kx^{k-1} J_k(x) = x^k J_{k-1}(x)$

dividing through by x^k:

$\Rightarrow J_k'(x) + \frac{k}{x} J_k(x) = J_{k-1}(x)$ ——— (1)

Similarly ~~from~~ (ii) $\frac{d}{dx}\left[x^{-k} J_k(x)\right] = -x^{-k} J_{k+1}(x)$

~~dividing through by x^{-k}~~

$x^{-k} J_k'(x) - kx^{-k-1} J_k(x) = -x^{-k} J_{k+1}(x)$

diving through by x^{-k}

$\Rightarrow J_k'(x) - \frac{k}{x} J_k(x) = -J_{k+1}(x)$ ——— (2)

Equation (1) + equation (2)

$\Rightarrow 2J_k'(x) = J_{k-1}(x) - J_{k+1}(x)$

$\therefore J_k'(x) = \frac{1}{2}\left[J_{k-1}(x) - J_{k+1}(x)\right]$

iv) $J_{k+1}(x) + J_{k-1}(x) = \frac{2k}{x} J_k(x)$

Equation (1) - equation (2)

$\frac{2k}{x} J_k(x) = J_{k-1}(x) + J_{k+1}(x)$

If $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ find $J_{3/2}(x)$

Using $J_{k+1}(x) = \frac{2k}{x} J_k(x) - J_{k-1}(x)$

$k = 1/2 \Rightarrow J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

$$J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \frac{1}{x}\cdot\sqrt{\frac{2}{\pi x}}\left[\sin x - \cos x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x} - \cos x\right]$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x - x\cos x}{x}\right] //$$

* Generating functions

The generating function for the bessel's polynomials is :

$$e^{\frac{x}{2}(t - 1/t)} = \sum_{n=0}^{\infty} J_n(x) t^n = J_0(x) + J_1(x)t + J_2(x)t^2 + \cdots J_n$$

$$\Rightarrow e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}} = \left[1 + \frac{xt}{2} + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{xt}{2}\right)^n + \cdots\right]$$

$$\left[1 - \frac{x}{2t} + \frac{1}{2!}\left(\frac{-x}{2t}\right)^2 + \frac{1}{3!}\left(-\frac{x}{2t}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{-x}{2t}\right)^n + \cdots\right]$$

for $J_0 =$ $\downarrow\downarrow\downarrow\downarrow$

$J_1 =$ $\swarrow\swarrow\swarrow\swarrow$

$$= \left[1 + \frac{xt}{2} + \frac{x^2t^2}{2!\,2^2} + \frac{x^3t^3}{3!\,2^3} + \cdots \frac{x^n t^n}{2^n n!} + \cdots\right]$$

$$\left[1 - \frac{x}{2t} + \frac{x^2}{2^2 2!\, t^2} - \frac{x^3}{2^3 3!\, t^3} + \cdots \frac{(-1)^n x^n}{2^n t^n n!} + \cdots\right]$$

$1 \times 1 = 1$

$\frac{xt}{2} \times \frac{x}{2t} = \frac{x^2}{2}$

We now multiply term by term

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{2^4\cdot(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots + \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \pm \cdots$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\cdot(n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{2^4(2!)^2} - \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\cdot 2!} + \frac{x^5}{2^5 2!3!}$$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{1+2n}}{2^{1+2n}\cdot n!(n+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} + \cdots +$$

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Legendre polynomials

The Legendre polynomials are obtained from the solution of the legendre differential Equation $(1-x^2)y'' - 2xy + p(p+1)y = 0$ where p is a real number.

Example

$$(1-x^2)y'' - 2xy' + 2y = 0, \quad p = 1$$

Soln: Clearly $x = \pm 1$ are singular points, accordingly let the solution of the differential Equation be

$$y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}, \quad y'(x) = m+n \sum_{n=0}^{\infty} a_n x^{m+n-1}$$

$$y''(x) = (m+n-1)(m+n) \sum_{n=0}^{\infty} a_n x^{m+n-2}$$

substituting into the legendre equation gives

$$(m+n)(m+n-1)\sum_{n=0}^{\infty} a_n x^{m+n-2} - (m+n)(m+n-1)\sum_{n=0}^{\infty} a_n x^{m+n}$$
$$-2(m+n)\sum_{n=0}^{\infty} a_n x^{m+n} + p(p+1)\sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

The Indicial Equation is the least power of x. we go with the coefficient (of) $n=0$

$$(m+n)(m+n-1) = 0$$

for $n=0$ we $(m+0)(m+0-1) = 0$

$$m(m-1) = 0 \quad \therefore m=0, m=1$$

~~Again~~

$$(m+n+2)(m+n+1)a_{n+2} - (m+n)(m+n-1)a_n - 2(m+n)a_n + p(p+1)a_n$$
$$(m+n+2)(m+n+1)a_{n+2} = (m+n)(m+n-1)a_n + 2(m+n)a_n - p(p+1)a_n$$
$$(m+n+2)(m+n+1)a_{n+2} = [(m+n)(m+n-1) + 2 - p(p+1)]a_n$$
$$(m+n+2)(m+n+1)a_{n+2} = [(m+n)(m+n+1) - p(p+1)]a_n$$
$$\therefore a_{n+2} = \frac{[(m+n)(m+n+1) - p(p+1)]a_n}{(m+n+2)(m+n+1)}, \quad n \geq 0$$

Case 1 $m=0$:

$$a_{n+2} = \frac{[n(n+1) - p(p+1)]a_n}{(n+2)(n+1)}$$

$n=0$: $a_2 = \dfrac{-p(p+1)a_0}{2}$

$n=1$: $a_3 = \dfrac{2-p(p+1)a_1}{2\cdot 3} = -\dfrac{(p^2+p-2)a_1}{2\cdot 3} = -\dfrac{((p-1)(p+2)a_1}{2\cdot 3}$

$n=2$: $a_4 = \frac{[2\cdot 3 - p(p+1)]}{3\cdot 4} a_2 = -\frac{(p+3)(p-2)}{3\cdot 4} a_2$

or $a_4 = \frac{p(p+1)(p-2)(p+3)}{2\cdot 3\cdot 4} a_0$

from $y(x) = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + a_4x^{m+4}$

but $m=0$

$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

$\therefore y(x) = a_0 + a_1x - \frac{p(p+1)}{2}a_0x^2 - \frac{(p-1)(p+2)}{6}a_1x^3$

$+ \frac{p(p+1)(p-2)(p+3)}{24}a_0x^4 - - -$

$\therefore y_1(x) = a_0\left[1 - \frac{p(p+1)}{2!}x^2 - \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + ---\right.$

$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + ---\right.$

NOTE THAT:

1. If $p \geq 0$ is an even integer the first part of the series terminates; and similarly
2. If $p \geq 0$ is an odd integer the second part of the series terminates.

furthermore if $n=p$. $a_{n+2} = a_{n+4} = 0$

(if $n = p-2, p-4$ be substitute into Recur

re $a_{n+2} = \frac{[n(n+1) - p(p+1)]}{(n+2)(n+1)} a_n$

$$a_p = \frac{[(p-2)(p-1) - p(p+1)]}{p(p-1)} a_{p-2}$$

$$= \frac{[p^2 - 3p + 2 - p^2 - p]}{p(p-1)} a_{p-2} \quad \text{or } a_p = \frac{(-4p+2)a_{p-2}}{p(p-1)}$$

$$a_p = \frac{-2(2p-1)a_{p-2}}{p(p-1)}$$

$$\therefore a_{p-2} = \frac{-p(p-1)a_p}{2(2p-1)}$$

for $n = p-4$

$$a_{p-2} = \frac{[(p-4)(p-3) - p(p+1)]}{(p-3)(p-2)} a_{p-4} = a_{n+2} =$$

$$= \frac{[p^2 - 7p + 12 - p^2 - p]}{(p-3)(p-2)} a_{p-4} = \frac{-4(2p-3)a_{p-4}}{(p-2)(p-3)}$$

$$\therefore a_{p-4} = \frac{-(p-2)(p-3)}{4(2p-3)} a_{p-2} = \frac{p(p-1)(p-2)(p-3)}{2\cdot 4(2p-1)(2p-3)} a_p$$

$$\therefore y_1(x) = a_p x^n - \frac{p(p-1)x^{n-2}}{2(2p-1)} a_p + \frac{p(p-1)(p-2)(p-3)}{2\cdot 4(2p-1)(2p-3)} a_p x^{n-4}$$

$$\text{or } y_1(x) = a_p \left[x^n - \frac{p(p-1)}{2(2p-1)} x^{n-2} + \frac{p(p-1)(p-2)(p-3)}{2\cdot 4(2p-1)(2p-3)} x^{n-4} + \cdots\right.$$

Using $A_p = \dfrac{(2p-1)(2p-3 \cdots 3\cdot 1}{p!}$

we have

$$y_p(x) = \frac{(2p-1)(2p-3)\cdots 3\cdot 1}{p!}\left[x^p - \frac{p(p-1)}{2(2p-1)} x^{p-2} \cdots\right.$$

Or

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}\left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} + \cdots\right]$$ (*)

note that: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$, $P_4(x) =$

Rodrigues formula

If $$\boxed{P_k(x) = \frac{1}{2^k\cdot k!}\frac{d^k}{dx^k}(x^2-1)^k}$$

* Generating function: The generating function for the Legendre polynomial is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

$$J_k(x) = e^{\frac{x}{2}(t-1/t)}$$

note that: If $P_n(x)$ is the Legendre polynomial of order n use the generating function to show that

$$P_{n+1}(x) = \frac{(2n+1)}{n+1}x P_n(x) - \frac{n}{n+1}P_{n-1}(x)$$

If $P_0(x) = 1$, $P_1(x) = x$

find $P_2(x)$ and $P_3(x)$.

Solution

from $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

$\dfrac{1}{\sqrt{1-2xt+t^2}} \Rightarrow (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$

Differentiate with for t.

$\Rightarrow \dfrac{-\frac{1}{2}(2t-2x)}{(1-2xt+t^2)^{3/2}} = n\sum_{n=0}^{\infty} P_n(x)\, t^{n-1}$

$\Rightarrow \dfrac{x-t}{\sqrt{(1-2xt+t^2)}} = n\sum_{n=0}^{\infty} P_n(x)\, t^{n-1}$

multiply through by $(1-2xt+t^2)$

$\therefore \dfrac{(x-t)}{\sqrt{(1-2xt+t^2)}} = n\sum_{n=0}^{\infty} P_n(x)\, t^{n-1}\,(1-2xt+t^2)$

$(x-t)\sum_{n=0}^{\infty} P_n(x) t^n = n\sum_{n=0}^{\infty} P_n(x)\, t^{n-1} - 2nx\sum_{n=0}^{\infty} P_n(x) t^n + n\sum_{n=0}^{\infty} P_n(x) t^{n+1}$

$\sum_{n=0}^{\infty}\left[x P_n(x) t^n - P_{n-1}(x) t^n\right] = \sum_{n=0}\left[(n+1) P_{n+1}(x) t^n - 2nx\, P_n(x) t^n + (n-1) P_{n-1}(x) t^n\right]$

$x P_n(x) t^n - P_{n-1}(x) t^n = (n+1) P_{n+1}(x) t^n - 2nx\, P_n(x) t^n + (n-1) P_{n-1}(x) t^n$

$(x P_n(x) - P_{n-1}(x)) t^n = \left[(n+1) P_{n+1}(x) - 2nx\, P_n(x) + (n-1) P_{n-1}(x)\right] t^n$

$(n+1) P_{n+1}(x) = (2n+1)\, x P_n(x) - n P_{n-1}(x)$

$\therefore P_{n+1}(x) = \dfrac{2n+1}{n+1}\, x P_n(x) - \dfrac{n}{n+1} P_{n-1}(x)$

find $P_2(x)$ and $P_3(x)$ if $P_0(x)=1$ $P_1(x)=x$

$\therefore P_2(x) = \frac{3}{2}\cdot x\cdot x - \frac{1}{2}\cdot 1 = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{5}{2}x P_2(x) - \frac{2}{3}P_1(x) = \frac{5}{3}x\cdot\frac{1}{2}(3x^2-1) - \frac{2}{3}x$

* **Example**

If P_m and P_n are legendre polynomials of Order m and n. show that

(1) $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$, if $m \neq n$

(2) $\int_{-1}^{1} (P_n(x))^2\,dx = \frac{2}{2n+1}$

proof

~~from~~ $P_m(x)$ and $P_n(x)$ are legendre polynomials then they satisfy the legendre Equation:

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$\Rightarrow (1-x^2)P_m''(x) - 2xP_m' + m(m+1)P_m = 0$ ——— (1)

$(1-x^2)P_n''(x) - 2xP_n' + n(n+1)P_n = 0$ ——— (2)

multiplying equ (1) by $P_n(x)$ and equation (2) by $P_m(x)$

$\Rightarrow (1-x^2)P_m''P_n(x) - 2xP_m'P_n(x) + m(m+1)P_mP_n = 0$ ——— (3)

$\Rightarrow (1-x^2)P_n''P_m(x) - 2xP_n'P_m(x) + n(n+1)P_nP_m = 0$ ——— (4)

Equation (3) − equation (4)

$(1-x^2)\left[P_m''P_n(x) - P_n''P_m(x)\right] - 2x\left\{P_m'P_n(x) - P_n'P_m(x)\right\} + \left\{m(m+1) - n(n+1)\right\}P_nP_m(x) = 0$

$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\left(P_nP_m' - P_n'P_m\right)\right\} + \left(m(m+1) - n(n+1)\right)P_nP_m(x) = 0$

Integrating both side

$$\Rightarrow (1-x^2)\int_{-1}^{1}(P_n P_m' - P_m P_n') = n(n+1) - m(m+1)\int_{-1}^{1} P_n(x) P_m(x)\, dx$$

$$\Rightarrow (n(n+1) - m(m+1)) \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0$$

Clearly if $n \neq m$

$$\Rightarrow n(n+1) - m(m+1) \neq 0$$

$$\Rightarrow \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0, \quad n \neq m.$$

(2) To show that

$$\int_{-1}^{1} (P_n(x))^2\, dx = \frac{2}{2n+1}$$

sol

from generating function $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$

squaring both side we have

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} (P_n(x))^2 t^{2n}$$

Integrating both side from (-1) to (1) wrt x.

$$\int_{-1}^{1} \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} \int_{-1}^{1} (P_n(x))^2 t^{2n}\, dx$$

$$-\frac{1}{2t}\ln(1-2xt+t^2)\Big|_{-1}^{1} = \int_{-1}^{1} (P_n(x))^2 t^{2n}\, dx$$

$$\Rightarrow -\frac{1}{2t}\left\{\ln(1-2t+t^2) - \ln(1+2t+t^2)\right\} = \int_{-1}^{1} ((P_n(x))^2 t^{2n}\, dx$$

$$\frac{1}{2t}\left\{\ln(1+t)^2 - \ln(1-t)^2\right\} = \int_{-1}^{1} (P_n(x))^2 t^{2n}\, dx$$

$$\ln(1+t) - \ln(1-t) = \int_{-1}^{1} (P_n(x))^2 t^{2n+1}\, dx$$

$$= \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} + \right) = \int_{-1}^{1} (P_n(x))^2 t^{2n+1} dx$$

$$= 2t + \frac{2t^3}{3} + \cdots = \int_{-1}^{1} (P_n(x))^2 t^{2n+1} dx$$

$$= 2 + \frac{2t^2}{3} + \cdots \int_{-1}^{1} (P_n(x))^2 \cdot t^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{2t^{2n}}{2n+1} = \sum_{n=0}^{\infty} \int_{-1}^{1} (P_n(x))^2 t^{2n} dx$$

$$\Rightarrow \frac{2}{2n+1} = \int_{-1}^{1} (P_n(x))^2 dx.$$

30/3/23

Initial value problem & Boundary value problem

Definition :- An Initial value problem (IVP) is a differential equation together with the associated Initial Conditions.

Example

a) $\frac{dy}{dx} = x + y$, $y(0) = 1$,

b) $y'' + 2xy' + y = 0$ $y(0) = 0$, $y(0) = -1$

The general form of an I.V.P is given by

$$y^{(n)} + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_0 y = f(t)$$

$$y(t_0) = y_0,\ y'(t_0) = y_1,\ y''(t_0) = y_2 \ldots y^{n-1}(t_0) = y_{n-1}$$

Definition : A boundary value problem (B.V.P) is a differential equation together with the associated boundary Conditions.

a) $y'' + 7y = 0, \quad y(0) = 0, \quad y'(\pi) = 0 \qquad 0 \le x \le \pi$

b) $y'' + 2xy' + y = 0, \quad y(0) = 1, \quad y(2) = 4 \qquad 0 \le x \le 2$

THEOREM *

Existence and Uniqueness (of solution)

Given the I.V.P

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

(a) Let $f(x,y)$ be continuous and bounded in the region R, given by R:

$$|x - x_0| \le a, \quad |y - y_0| \le b$$

Then the I.V.P has at least one solution in the interval $|x - x_0| \le h$, where $h = \min(a, b/M)$ where $|f(x,y)| \le M$.

(b) Furthermore, if the partial derivative $\frac{\partial f}{\partial y}$ is continuous and bounded in R, then the I.V.P has a unique solution in the interval $|x - x_0| \le h$ $\square$.

To Check for Continuous of the fxn:

i) If the fxn is a linear fxn, then it is continuous e.g. $\frac{dy}{dx} = x+y$, $\frac{dy}{dx} = x^2y + y^2$

ii) If the fxn is quotient fxn, then we check: i.e. if the denominator is containing x (we use interval of values of x) & if it is y (we use interval of values of y).

iii) Every constant fxn is continuous.

Example 1

Given the initial value problem

$$\frac{dy}{dx} = x + y \quad y(0) = 1, \quad R: |x| \le 1,$$

$|y - 1| \le 2$, show that the I.V.P has a unique solution in R.

Solution $R: |x| \leq 1 \Rightarrow x_0 = 0$

$|y-1| \leq 2 \Rightarrow y_0 = 1$

$\frac{dy}{dx} = x+y$, $y(0) = 1$, $f(x,y) = x+y$

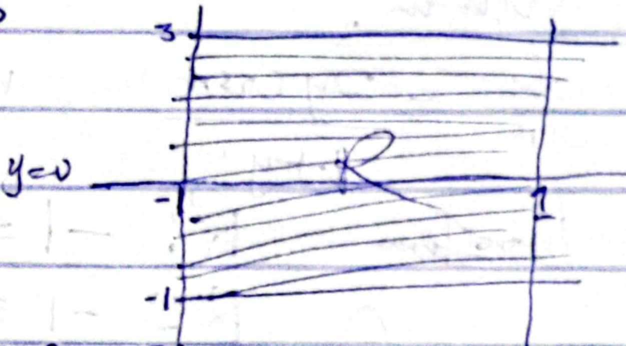
where $x_0 = 0$ & $y = 1$ $y_0 = 1$

$R: |x - x_0| \leq a \Rightarrow |x - 0| \leq 1 \Rightarrow |x| \leq 1$

$\Rightarrow -1 \leq x \leq 1$ and

$|y - y_0| \leq b \Rightarrow |y-1| \leq 2 \Rightarrow -2 \leq y-1 \leq 2$

$\Rightarrow -1 \leq y \leq 3$



It is clear that $f(x,y) = x+y$ is continuous

$f(x,y)$ is bounded. If $|f(x,y)| \leq M$

$\Rightarrow |f(x,y)| = |x+y| \leq |x|+|y| = 1+3 = 4 \therefore M=4$

Hence the I.V.P has at least one solution in the interval $|x - x_0| \leq h$, therefore

$h = \min(a, b/M) = \min(1, 2/4) = (1, 1/2) = 1/2$

i.e the I.V.P has solution in the interval $|x| \leq 1/2$ & $(-1/2 \leq x \leq 1/2)$.

Furthermore, $f(x,y) = x+y \Rightarrow \frac{\partial f}{\partial y} = 1$.

Since the partial derivative $\frac{\partial f}{\partial y}$ is continuous in R, then the I.V.P has a unique solution in R.

Example 2

Use the existence and uniqueness theorem to show that the I.V.P

$$\frac{dy}{dx} = \frac{2y\cos x}{1+y^2}, \quad y(0)=1$$ has a

solution in the region $R: |x-0| \le 1$ & $|y-1| \le 2$

solution

$$\frac{dy}{dx} = \frac{2y\cos x}{1+y^2}, \quad y(0) = 1$$

therefore $R: -1 \le x \le 1$

$R: -1 \le y \le 3$

Clearly $f(x,y) = \frac{2y\cos x}{1+y^2}$ is continuous in R

$$|f(x,y)| = \frac{|2y\cos x|}{1+y^2} \le 2\frac{|y||\cos x|}{1+y^2}$$

$$\le 2\cdot|y||\cos x| = 2\cdot 3\cdot 1 = 6$$

$$\Rightarrow \frac{|y|}{1+y^2} \le y \Rightarrow |f(x,y)| \le 6.$$

Hence the I.V.P has atleast one solution in the interval $|x-x_0| \le h$, $h = \min(1, 3/6)$

$= \min(1, 1/2) = 1/2$

i.e $|x| \le 1/2$

furthermore, $\frac{\partial f}{\partial y} = \frac{(1+y^2)\,2\cos x - 2y(2y\cos x)}{(1+y^2)^2}$

$$= \frac{2(1-y^2)\cos x}{(1+y^2)^2}$$

Since $\frac{\partial f}{\partial y}$ is continuous in R, then the I.v.p has a unique solution.

Exercises

Discuss the existence and uniqueness of solution for the I.v.p

not cont. a) $\frac{dy}{dx} = \frac{3xy^2}{1-x^2}$, $y(0)=1$ then the rectangle $R: |x| \le 2,\ |y-1| \le 1$

cont. b) $\frac{dy}{dx} = \frac{3xy^2}{1+x^2}$, $y(0)=1$
$R: |x| \le 1,\ |y-1| \le 2$

Continuous c) $\frac{dy}{dx} = xy^2 + y^2$, $y(1) = 1$,
$R: |x-1| \le 1,\ |y-1| \le 2.$

3/4/2023

System of simultaneous diff. Equation

Examples: solve the system of Equations

1) $\frac{dx}{dt} + y = e^t$, $x - \frac{dy}{dt} = t$, $x = x(t)$
$y = y(t)$

The above system can be expressed in matrix form as:

$$A\tilde{X} = T \Rightarrow A = \begin{bmatrix} \frac{d}{dt} & 1 \\ 1 & -\frac{d}{dt} \end{bmatrix}, \tilde{X} = \begin{bmatrix} X \\ Y \end{bmatrix}; T = \begin{bmatrix} e^t \\ t \end{bmatrix}$$

or $$\begin{bmatrix} \frac{d}{dt} & 1 \\ 1 & -\frac{d}{dt} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Accordingly $X = \frac{\Delta x}{\Delta}$, $Y = \frac{\Delta_Y}{\Delta}$

$$\Rightarrow \frac{\begin{vmatrix} e^t & 1 \\ t & -\frac{d}{dt} \end{vmatrix}}{\begin{vmatrix} \frac{d}{dt} & 1 \\ 1 & -\frac{d}{dt} \end{vmatrix}} = X \Rightarrow \frac{-\frac{d}{dt}(e^t) - t}{-\frac{d^2}{dt^2} - 1} = X$$

$$\Rightarrow -e^t - t = \left(-\frac{d^2}{dt^2} - 1\right) X$$

$$\Rightarrow \frac{d^2 X}{dt^2} + X = e^t + t \quad \text{—— (1)}$$

Similarly $Y = \frac{\Delta y}{\Delta}$

$$\Rightarrow \frac{\begin{vmatrix} \frac{d}{dt} & e^t \\ 1 & t \end{vmatrix}}{\begin{vmatrix} \frac{d}{dt} & 1 \\ 1 & -\frac{d}{dt} \end{vmatrix}} = Y \Rightarrow \frac{\frac{d}{dt}(t) - e^t}{-\frac{d^2}{dt^2} - 1} = Y$$

$$\Rightarrow \frac{d}{dt}(t) - e^t = \left(-\frac{d^2}{dt^2} - 1\right) Y$$

$$\therefore \frac{d^2 Y}{dt^2} + Y = e^t - 1 \quad \text{--- (2)}$$

from equation (1)

The auxiliary equation is $m^2+1=0$

$\therefore \quad m = \pm i$

$x_c(t) = c_1 \cos t + c_2 \sin t$

let $x_p = ae^t + bt + c$

$x_p' = ae^t + b$

$x_p'' = ae^t$

$\therefore \quad ae^t + ae^t + bt + c = e^t + t$

By comparing $a = 1/2, \quad b = 1 \quad c = 0$

$\therefore \quad x_p = \frac{1}{2}e^t + t$

$\therefore \quad x(t) = c_1 \cos t + c_2 \sin t + \frac{1}{2}e^t + t$

Again from equation (2)

$y_c(t) = c_3 \cos t + c_4 \sin t$

let $y_p = de^t + k, \quad y_p' = de^t, \quad y'' = de^t$

$de^t + de^t + k = e^t - 1$

$\Rightarrow \quad 2d = 1$ or $d = 1/2, \quad k = -1$

$\Rightarrow \quad y_p = \frac{1}{2}e^t - 1$

$\therefore \quad y_p(t) = c_3 \cos t + c_4 \sin t + \frac{1}{2}e^t - 1$

2) $\frac{dx}{dt} + 4x + \frac{dy}{dt} = 1$, $\frac{dx}{dt} - 2x + y = t^2$

solution

$(D+4)x + Dy = 1$, $(D-2)x + D = t^2$

where $D = d/dt$

$$\begin{bmatrix} D+4 & D \\ D-2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ t^2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} D+4 & D \\ D-2 & 1 \end{vmatrix}$$

$(D+4) - [D(D-2)] = D+4 - D^2 + 2D$

$\therefore \Delta = -D^2 + 3D + 4$

$$\Delta x = \begin{vmatrix} 1 & D \\ t^2 & 1 \end{vmatrix} = 1 - D(t^2) = 1 - 2t$$

$\therefore \frac{\Delta x}{\Delta} = x \Rightarrow \frac{1-2t}{-D^2+3D+4} = x$

or $(-D^2+3D+4)x = 1-2t$

$\therefore (D^2-3D-4)x = 2t-1 \longrightarrow (1)$

Similarly $y = \frac{\Delta y}{\Delta}$

$$\Delta y = \begin{vmatrix} D+4 & 1 \\ D-2 & t^2 \end{vmatrix} = (D+4)t^2 - (D-2)(1)$$

$\Delta y = 2t + 4t^2 + 2$

$\therefore \frac{\Delta y}{\Delta} = y \Rightarrow \frac{2t+4t^2+2}{-D^2+3D+4} = y$

auxiliary (x_c) particular (x_p)

$$\Rightarrow (D^2 - 3D - 4)y = 4t^2 - 2t - 2 \longrightarrow (2)$$

from equation (1),

The auxiliary equation is

$$m^2 - 3m - 4 = 0$$

$$\therefore m = -1 \;\&\; 4$$

$$x_c(t) = c_1 e^{-t} + c_2 e^{4t}$$

let $x_p = at^2 + bt + c$

$$x'_p = 2at + b,$$

$$x''_p = 2a$$

$$\therefore 2a - 6at - 3b - 4at^2 - 4bt - 4c = 2t - 1$$

$$\therefore 2a - 3b - 4c = -1$$

$$-6a - 4b = 2$$

$$-4a = 0$$

$$\therefore a = 0, \; a = 0 \Rightarrow -4b = 2 \therefore b = -1/2$$

$$2a - 3b - 4c = -1 \Rightarrow -3(-1/2) - 4c = -1$$

$$\therefore -4c = -1 - 3/2 \therefore c = 5/8$$

$$x_p = -1/2\, t + 5/8$$

$$x(t) = c_1 e^{-t} + c_2 e^{4t} - 1/2\, t + 5/8$$

from equation (2)

The auxiliary equation is

$$y_c(t) = c_3 e^{-t} + c_4 e^{4t}$$

let $y_p(t) = at^2 + bt + c$

$$y'_p(t) = 2at + b,$$

$$y''_p(t) = 2a$$

$2a-3b-4c=-2$

$-6a-4b=-2$

$-4a=-4 \quad \therefore \underline{a=1}$

$-6-4b=-2 \Rightarrow -4b=4 \quad \therefore b=-1$

$2-3(-1)-4c=-2$

$-4c=-7 \quad \therefore c=7/4$

$y_p=t^2-t+7/4$

$\therefore y(t)=c_3e^{-t}+c_4e^{4t}+t^2-t+7/4$

Exercise

$4\frac{dx}{dt}-\frac{dy}{dt}+3x=\sin t$

$\frac{dx}{dt}+y=\cos t \quad x=0,\ y=0$ when $t=0$

Example ③ Higher order

Solve the system of Equations

$\frac{d^2x}{dt^2}+\frac{dy}{dt}+3x=e^{-t}, \quad \frac{d^2y}{dt^2}-4\frac{dx}{dt}+3y=\sin 2t$

Soln

$(D^2+3)x+Dy=e^{-t}, \quad -4Dx+(D^2+3)y=\sin 2t$

where $Dx=\frac{dx}{dt}, \quad Dy=\frac{dy}{dt}$

$\therefore A\tilde{x}=T \Leftrightarrow \begin{bmatrix} D^2+3 & D \\ -4D & D^2+3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} e^{-t} \\ \sin 2t \end{bmatrix}$

$\Delta=(D^2+3)^2-D(-4D)=D^4+6D^2+9+4D^2=D^4+10D^2+9$

$\bar{X} = \begin{bmatrix} X \\ Y \end{bmatrix}$ ∴ $X = \frac{\Delta_x}{\Delta}$ and $Y = \frac{\Delta_y}{\Delta}$

$$X = \frac{\begin{vmatrix} e^{-t} & D \\ \sin 2t & D^2+3 \end{vmatrix}}{\begin{vmatrix} D^2+3 & D \\ -4D & D^2+3 \end{vmatrix}} = \frac{(D^2+3)e^{-t} - D(\sin 2t)}{D^4+10D^2+9} = X$$

$\Rightarrow (D^2+3)e^{-t} - D\sin 2t = X(D^4+10D^2+9)$

$\Rightarrow (D^4+10D^2+9)X = 4e^{-t} - 2\cos 2t \quad \text{--- (1)}$

Similarly $Y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} D^2+3 & e^{-t} \\ -4D & \sin 2t \end{vmatrix}}{\Delta}$

$\Rightarrow \frac{(D^2+3)\sin 2t + 4De^{-t}}{D^4+10D^2+9} = Y$

$\Rightarrow (D^4+10D^2+9)Y = -4\sin 2t + 3\sin 2t - 4e^{-t}$

$(D^4+10D^2+9)Y = -\sin 2t - 4e^{-t} \quad \text{--- (2)}$

from (i) $D^4+10D^2+9 = 4e^{-t} - 2\cos 2t$

∴ $D^4+10D^2+9 = 0 \Rightarrow$ The auxiliary Equation

∴ $m^4 + 10m + 9 = 0$

let $m^2 = k \Rightarrow k^2 + 10k + 9 = 0$

∴ $(k+1)(k+9) = 0 \quad k=-1, k=-9$

∴ $m^2 = -1 \Rightarrow m = \pm\sqrt{-1} = \pm i$,

$m^2 = -9 \Rightarrow m = \pm\sqrt{-9}$ or $m = \pm 3i$

The Comp. solution is $X(t) = C_1\cos t + C_2\sin t + C_3\cos 3t + C_4\sin 3t$

(i) If $f(x) = e^{px}$ then $\frac{1}{\phi(D)} e^{px} = \frac{e^{px}}{\phi(p)}$

(ii) If $f(x) = \cos px$ or $\sin px$ then $\frac{1}{\phi(D^2)} \sin px = \frac{1}{\phi(-p^2)} \sin px$

$\frac{1}{\phi(D)} f(x)$

$f(D^2) = \frac{1}{\phi(-p^2)}$

Again $y_p = \frac{1}{D^4+10D^2+9}\left(4e^{-t} - 2\cos 2t\right)$

$$= \frac{1}{D^4+10D^2+9}\, 4e^{-t} - 2\cdot\frac{1}{D^4+10D^2+9}\cdot\cos 2t$$

$$y_p = \frac{4e^{-t}}{(-1)^4+10(-1)^2+9} - \frac{2\cos 2t}{(-2^2)^2+10(-2^2)+9}$$

$$y_p = \frac{4e^{-t}}{20} - \frac{2\cos 2t}{-15} \quad \therefore\ y_p = \frac{e^{-t}}{5} + \frac{2\cos 2t}{15}$$

~~[illegible]~~:

$$X(t) = C_1\cos t + C_2\sin t + C_3\cos 3t + C_4\sin 3t + \frac{e^{-t}}{5} + \frac{2\cos 2t}{15}$$

Similarly:

$$Y_c = C_5\cos t + C_6\sin t + C_7\cos 3t + C_8\sin 3t$$

$$Y_p = \frac{1}{D^4+10D^2+9}\left(-\sin 2t - 4e^{-t}\right)$$

$$= \frac{1}{D^4+10D^2+9}(-\sin 2t)\ .\ \frac{1}{D^4+10D^2+9}\left(-4e^{-t}\right)$$

$$= \frac{-\sin 2t}{(-2^2)^2+10(-2^2)+9} - \frac{4e^{-t}}{(-1)^4+10(-1)^2+9}$$

$$= \frac{-\sin 2t}{16-40+9} - \frac{4e^{-t}}{1+10+9} \Rightarrow$$

$$Y_p = \frac{\sin 2t}{15} - \frac{e^{-t}}{5}$$

$$\therefore Y(t) = C_5\cos t + C_6\sin t + C_7\cos 3t + C_8\sin 3t + \frac{\sin 2t}{15} - \frac{e^{-t}}{5}$$

$y_p = \frac{1}{2D-1}\sin 2x = \frac{(2D+1)\sin 2x}{(2D-1)(2D+1)} = \frac{(2D+1)\sin 2x}{4D^2-1} = \frac{(2D+1)\sin 2x}{4(-2^2)-1} = \frac{(2D+1)\sin 2x}{-17}$

$y_p = \frac{\sin 2x}{-4+2D+3}$

$y_p = \frac{1}{D^2+2}e^{3x} =$

$y_p = \frac{1}{D^2+2}\sin 3x = \frac{\sin 3x}{(-3^2)+2} = \frac{\sin 3x}{-7}$

Exercise: solve the Equation-

1) $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$, $2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$

$x = 0$, $y = 0$, $\frac{dy}{dt} = 0$ and $\frac{dy}{dt} = 0$, when $t = 0$.

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* LIPSCHITZ CONDITIONS (FOR I.V.P)

Definition:

~~A~~ function $f(x,y)$ defined and continuous in a domain D of the xy-plane (is said) to satisfy the Lipschitz (or is Lipschitz continuous on y) if there exist a constant K such that

$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$ $\forall$ points (x,y_1) & (x,y_2) in D.

$y' = f(x,y)$, $y(x_0) = y_0$

Existence: If $f(x,y)$ is continuous and bounded in $R: |x - x_0| \leq a$, $|y - y_0| \leq b$, then $\exists$ at least one solution exist in the interval $|x - x_0| < h$, $h = \min(a, b/M)$

Uniqueness: If $\frac{\partial f}{\partial y}$ is continuous and bounded in R then the I.V.P has at most one solution.

Example

Show that the functions satisfy the Lipschitz condition in the indicated region.

1) $f(x,y) = x+2y$, $|x| \le 1$, $|y| \le 1$.

2) $f(x,y) = x+y^2$, $|x| \le 1$, $|y| \le 1$

Soln

① $f(x,y) = x+2y$ $R: |x| \le 1, |y| \le 1$

$$|f(x,y_1) - f(x,y_2)| = |x+2y_1 - (x+2y_2)|$$
$$= |x+2y_1 - x - 2y_2|$$
$$= |2y_1 - 2y_2|$$
$$\le 2|y_1 - y_2|, \quad K = 2$$

② $$|f(x,y_1) - f(x,y_2)| = |x+y_1^2 - (x+y_2^2)|$$
$$= |x+y_1^2 - x - y_2^2|$$
$$= |y_1^2 - y_2^2|$$
$$\le |y_1 + y_2||y_1 - y_2|$$
$$\le |1+1||y_1 - y_2|$$
$$\le 2|y_1 - y_2| \Rightarrow K = 2$$

Note that: from the mean value theorem

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(y_1 - y_2)$$

$$|f(x,y_1) - f(x,y_2)| \le \left|\frac{\partial f}{\partial y}\right||y_1 - y_2| \quad \therefore \frac{\partial f}{\partial y} \le K$$
$$\le K|y_1 - y_2|$$

Uniqueness: If $f(x,y)$ in addition satisfies the Lipschitz's Condition in the rectangle R. Then the I.V.P has a unique solution.

Example

Consider the I.V.P: $\frac{dy}{dx} = x^2y + y^2$, $y(0) = 1$

$R: |x-0| \le 2$, $|y-1| \le 2$.

(a) Show that $f(x,y)$ satisfies the lipschitz condition in R and find the lipschitz constant.

b) Determine whether of the I.V.P has a Unique solution in R.

(c) find the maximum value of $f(x,y)$ in R.

Solution $f(x,y) = x^2y + y^2$

(a) from $R: |x-0| \le 2 \Rightarrow |x| \le 2$

$R: |y-1| \le 2 \Rightarrow -2 \le y-1 \le 2 \Rightarrow -1 \le y \le 3$

$$
\begin{aligned}
|f(x,y_1) - f(x,y_2)| &= |x^2y_1 + y_1^2 - (x^2y_2 + y_2^2)| \\
&= |x^2y_1 + y_1^2 - x^2y_2 - y_2^2| \\
&\le |x^2(y_1 - y_2) + y_1^2 - y_2^2| \\
&\le |x|^2 |y_1 - y_2| + |y_1^2 - y_2^2| \\
&\le 2^2 |y_1 - y_2| + |y_1 + y_2| |y_1 - y_2| \\
&\le 4 |y_1 - y_2| + (|y_1| + |y_2|) |y_1 - y_2| \\
&\le 4 |y_1 - y_2| + (3+3) |y_1 - y_2| \\
&\le 4 |y_1 - y_2| + 6 |y_1 - y_2| \\
|f(x,y_1) - f(x,y_2)| &\le 10 |y_1 - y_2| \Rightarrow \underline{K = 10}
\end{aligned}
$$

(b) Given $y' = x^2y + y^2$, $f(x,y) = x^2y + y^2$

Clearly $f(x,y)$ is continuous in R

$f(x,y)$ is bounded in R if $\exists$ an $M \ni |f(x,y)| \leq M$

i.e $|f(x,y)| = |x^2y + y^2| \leq |x^2y| + |y^2|$

$\leq |x|^2|y| + |y|^2$

$= 2^2 \cdot 3 + 3^2$

$= 12 + 9 = 21$

$\Rightarrow M = 21$

$\therefore |f(x,y)| \leq 21 = M$. Hence $f(x,y)$ is bdd.

Since $f(x,y)$ is continuous and bounded in R

$\Rightarrow$ the IVP has atleast one solution in the

Interval: $|x - x_0| < h$, $h = \min(a, b/M) = (2, \frac{2}{21}) = \frac{2}{21}$

i.e $|x - x_0| \leq 2/21$

Since $f(x,y)$ is Lipschitz in R, then the IVP has a unique solution.

(c) $|f(x,y)| \leq 21 \Rightarrow$ The max. value of $f(x,y)$ is $\underline{21}$

Exercise

(1) Consider the I.V.P!: $\frac{dy}{dx} = 4x^2 + y^2$, $y(0) = 1$

$R: |x| \leq 1, |y| \leq 1$

a) Show that $f(x,y)$ satisfies the Lipschitz's condition in R and find the Lipschitz constant.

b) Determine if the I.V.P has a unique solution in R

c) find the max. value of $f(x,y)$ in R.

(2) Given the I.V.P : $y' = \frac{2y}{x}$, $y(1) = 1$, $|x-1| \leq 1/2$
$|y-1| \leq 1/2$

Determine if $f(x,y)$ satisfies the Lipschitz Condition in R. Hence discuss the existence and Uniqueness of solution for I.V.P in R.